Several or Multivariable Calculus Crash Course

For this "crash course" leading to Maxwell's Equations, we are going to extract a question from each section of the Stewart Calculus book that ties in with the desired vector calculus operations later on.

Chapter 12: Vectors and the Geometry of Space

12.1: Three-Dimensional Coordinate Systems

QUESTION [17] 12.1.43

Find the distance between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 4x + 4y + 4z - 11$.

ANSWER

$$2\sqrt{3} - 3$$

Step 1: The formulae from the section we are in:

Distance Formula Let $P(x_0, y_0, z_0),\ \ Q(x, y, z)$

$$|PQ| = \sqrt{(x - x_0)^2 + (y - y_o)^2 + (z - z_0)^2}$$

Equation of a Sphere Center: $C(h, k, l)$

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

Step 2: Analytical thinking: "distance between the spheres"

My first thoughts: (1) find the center points and the distance from them, (2) find the distance of each radius, add them together and then subtract them from the distance between the centers.

$$x^2 + y^2 + z^2 = 4 \ \Rightarrow\ (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = 2^2 \ \Rightarrow\ C(\mathbf{0}),\ \ r = 2.$$

Now, for sphere 2, complete the squares.

$$x^2 + y^2 + z^2 = 4x + 4y + 4z - 11 \ \Rightarrow\ x^2 + y^2 + z^2 - 4x - 4y - 4z = -11$$

$$\Rightarrow\ x^2 - 4x + y^2 - 4y + z^2 - 4z = -11$$

$$\Rightarrow\ -[x^2-4x+y^2-4y+z^2-4z]=\left(\sqrt{11}\right)^2$$

Completing the Square

$$ax^2+bx+c=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]=a\left[x^2+\frac{b}{a}x+\frac{c}{a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right]$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}\right].$$

We only have to do the completion of the square once as each of the squares are the same numerical value. I.e.,

$$x^2-4x=x^2-4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2=x^2-4x+4-4=(x-2)^2-4$$

$$\Rightarrow\ x^2-4x+y^2-4y+z^2-4z=-11$$

$$\Rightarrow\ ((x-2)^2-4)+((y-2)^2-4)+((z-2)^2-4)=-11$$

$$\Rightarrow\ (x-2)^2+(y-2)^2+(z-2)^2=-11+4+4+4$$

$$\Rightarrow\ (x-2)^2+(y-2)^2+(z-2)^2=1^2.$$

$$\therefore\ \ C(2,2,2),\qquad r=1.$$

Now, the distance between the center points are

$$|C_1C_2|=\sqrt{(2-0)^2+(2-0)^2+(2-0)^2}=\sqrt{12}=\sqrt{(3)(4)}=\sqrt{2^2(3)}=2\sqrt{3}.$$

The sum radii are $r_1+r_2=2+1=3$.

$$\therefore\ \ |C_1C_2|-r_1+r_2=2\sqrt{3}-3.$$

Step 3: Finalize the answer

Thus, this distance between the spheres is $2\sqrt{3}-3$ units.

NOTE Since we are referencing a distance, we would need to acknowledge units if in physics.

[RP] Prove that $2\sqrt{3}-3$ is the answer.